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RESOLVENTS AND BOUNDS FOR LINEAR AND NONLINEAR VOLTERRA EQUATIONS(1)

BY

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ABSTRACT. The asymptotic behavior of the resolvent of a linear Volterra equation is investigated without the assumption that the kernel of the equation is in $L^1(0, \infty)$. A lower bound is obtained on the solutions of a related nonlinear Volterra equation. A special case of the latter result is employed in the proof of the former result.

1. **Introduction.** For the linear Volterra equation

$$x(t) + \int_0^t a(t-s)x(s)\,ds = f(t) \qquad (0 \leq t < \infty), \tag{1.1}$$

where a and f are prescribed real valued functions, the resolvent (kernel) is defined to be the unique solution, r, of

$$r(t) + \int_0^t a(t-s)r(s)\,ds = a(t) \qquad (0 \leq t < \infty). \tag{1.2}$$

Thus r depends only on a and not on f.

There is a considerable literature dealing with properties and applications of the resolvent of (1.1). We shall comment on some of the earlier studies which are relevant to the present results. In Theorem 1.4 (below) the asymptotic behavior of r as $t \to \infty$ is investigated. Its proof employs several results which are relevant to nonlinear equations. In one of these, Theorem 1.7, a lower bound is obtained on the solutions of a nonlinear Volterra equation, (1.30) below, for which (1.1) is a special case.

The significance of the resolvent derives from the well known result:

LEMMA 1.1. *Let* $p \in [1, \infty]$ *and let*

$$a \in L^1_{\text{loc}}(0, \infty), \tag{1.3}$$

$$f \in L^p_{\text{loc}}(0, \infty). \tag{1.4}$$

Then the solution, x, *of* (1.1) *is given by*

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$$x(t) = f(t) - \int_0^t r(t-s) f(s)\, ds \qquad (0 \leq t < \infty) \tag{1.5}$$

and $x \in L^p_{\mathrm{loc}}(0, \infty)$.

The notation (1.4) means that

$$\int_0^T |f(t)|^p\, dt < \infty \quad \text{for all } T \in (0, \infty). \tag{1.6}$$

It is assumed that all prescribed functions are Lebesgue measurable and take values in the extended real line $[-\infty, \infty]$.

If x is a solution of the nonlinear Volterra equation

$$x(t) + \int_0^t a(t-s)\big[x(s) + h(x(s))\big]\, ds = F(t) \tag{1.7}$$

on some interval $0 \leq t < T \leq \infty$, where a, h, and F are prescribed, then Lemma 1.1 implies that x satisfies (1.5) on $[0, T)$ with

$$f(t) = F(t) - \int_0^t a(t-s) h(x(s))\, ds. \tag{1.8}$$

In the near linear case, $h(x) = o(x)$ as $x \to 0$, this reformulation of (1.7) as (1.5), (1.8) has been successfully employed in studying the asymptotic behavior of x as $t \to \infty$; see, e.g., Miller, Nohel, and Wong [6] and Nohel [7] and [8].

Some relevant facts concerning r are an immediate consequence of (1.1), (1.2), and Lemma 1.1:

LEMMA 1.2. (i) *If*

$$r \in L^1(0, \infty), \tag{1.9}$$

$$\lim_{t \to \infty} f(t) = f(\infty) \textit{ exists}, \tag{1.10}$$

then

$$\lim_{t \to \infty} x(t) = x(\infty) = f(\infty)\left[1 - \int_0^\infty r(t)\, dt\right], \tag{1.11}$$

where x is the solution of (1.1).

(ii) *If* (1.9) *and*

$$a \in L^1(0, \infty) \tag{1.12}$$

hold, then

$$\left[1 + \int_0^\infty a(t)\, dt\right] \int_0^\infty r(t)\, dt = \int_0^\infty a(t)\, dt. \tag{1.13}$$

(iii) *If* (1.9), (1.10), *and* (1.12) *hold, then*

$$\int_0^\infty a(t)\, dt \neq -1 \tag{1.14}$$

and

(1.15)
$$x(\infty) = \frac{f(\infty)}{1 + \int_0^\infty a(t)\,dt}.$$

Lemma 1.2 illustrates the usefulness of (1.9). Under hypothesis (1.12), this property is characterized in the classical result of Paley and Wiener [10, p. 60]:

THEOREM 1.1. *Let* (1.12) *hold. Then a necessary and sufficient condition for* (1.9) *to hold is that*

(1.16)
$$\int_0^\infty a(t)e^{-\lambda t}\,dt \neq -1 \quad \textit{for}\ \mathrm{Re}\,\lambda \geq 0.$$

Although (1.12) is not valid in some applications of (1.1), there are interesting alternative assumptions which imply (1.9). Theorem 1.2 below, due to Miller [5], is of this type. Also see [5] for references to specific physical problems which motivate these alternatives to (1.12).

THEOREM 1.2. *Let* (1.3) *and*

(1.17) $a \in C(0, \infty)$, *a is positive and nonincreasing on* $(0, \infty)$,

(1.18) $\dfrac{a(t)}{a(t+T)}$ *is nonincreasing on* $(0, \infty)$ *for each* $T > 0$

hold. Then

(1.19)
$$0 \leq r(t) \leq a(t) \qquad (0 < t < \infty),$$

(1.20)
$$\int_0^\infty r(t)\,dt = \frac{\int_0^\infty a(t)\,dt}{1 + \int_0^\infty a(t)\,dt} \quad \textit{if} \int_0^\infty a(t)\,dt < \infty,$$

(1.21)
$$\int_0^\infty r(t)\,dt = 1 \quad \textit{if} \int_0^\infty a(t)\,dt = \infty.$$

Since (1.16) is a consequence of (1.12) and (1.17), (1.20) also follows from Theorem 1.1 and Lemma 1.2; however, (1.19) and (1.21) are not contained in the preceding results. Miller [5] further shows that if a is completely monotonic, that is if

$$(-1)^k a^{(k)}(t) \geq 0 \qquad (k = 0, 1, \ldots; 0 < t < \infty),$$

and if $a(t) \not\equiv 0$, then (1.17) and (1.18) are satisfied.

A general result which does not assume (1.12) is the following one of Shea and Wainger [11]:

THEOREM 1.3. *Let*

(1.22)
$$a(t) = b(t) + c(t),$$

where

(1.23) $b \in L^1_{\mathrm{loc}}(0, \infty)$ *is nonnegative, nonincreasing, and convex on* $(0, \infty)$,

(1.24) $$c, tc \in L^1(0, \infty),$$

and let (1.16) *hold. Then* (1.9) *holds.*

When $a \not\in L^1(0, \infty)$ in Theorem 1.3, hypothesis (1.16) is to be interpreted for $\operatorname{Re} \lambda = 0$ as

$$\lim_{\sigma \to 0+} \int_0^\infty a(t) e^{-(\sigma + i\beta)t}\, dt \neq -1 \qquad (-\infty < \beta < \infty).$$

To show that the hypothesis of Theorem 1.3 with $c = 0$ is less restrictive than that of Theorem 1.2, let (1.17), (1.18), and $0 < t_1 < t_2$ be satisfied. Then

$$a(t_1) \Big/ a\left(\frac{t_1 + t_2}{2}\right) = a(t_1) \Big/ a\left(t_1 + \frac{t_2 - t_1}{2}\right)$$

$$\geqslant a\left(\frac{t_1 + t_2}{2}\right) \Big/ a\left(\frac{t_1 + t_2}{2} + \frac{t_2 - t_1}{2}\right) = a\left(\frac{t_1 + t_2}{2}\right) \Big/ a(t_2).$$

Thus

$$a\left(\frac{t_1 + t_2}{2}\right) \leqslant \left[a(t_1) a(t_2)\right]^{1/2} \leqslant \frac{a(t_1) + a(t_2)}{2}$$

and, hence, a is convex.

When $a(\infty) > 0$, Shea and Wainger [11, (6a) and (10)] have alternative assumptions to those of Theorem 1.3 which insure (1.9). Grossman and Miller [2, Theorems 5.1 and 5.2] have obtained fairly involved conditions insuring (1.9) when a is sufficiently smooth (or a sufficiently smooth perturbation of an L^1-function).

Here we prove:

THEOREM 1.4. *Let*

(1.25) a *be nonnegative and nonincreasing on* $(0, \infty)$,

(1.26) $$a(0) < \infty.$$

Then

(1.27) $$-a(0) \leqslant r(t) \leqslant a(0) \qquad (0 \leqslant t < \infty),$$

(1.28) $$0 \leqslant \int_0^t r(s)\, ds \leqslant 1 \qquad (0 \leqslant t < \infty),$$

(1.29) $$\lim_{t \to \infty} \int_0^t r(s)\, ds = \int_0^\infty r(s)\, ds \quad \textit{exists},$$

and (1.20), (1.21) *hold.*

In §5 the inequalities (1.27) are strengthened a little.

Note that $a \in C[0, \infty)$ is not assumed in Theorem 1.4. Both $a(\infty) > 0$ and $a(\infty) = 0$ are allowed. The example

$$\begin{aligned} &a(t) = 1 \quad (0 \leqslant t \leqslant 1), \quad a(t) = 0 \quad (1 < t < \infty), \\ &r(t) = e^{-t} \quad (0 \leqslant t \leqslant 1), \quad r(t) = e^{-t}[1 - 2e + et] \quad (1 < t \leqslant 2) \end{aligned}$$

shows, since $r(t) < 0$ for t in an interval to the right of 1, that (1.25) and (1.26) do not imply (1.19). A simple approximation argument now yields smooth $a(\cdot)$ satisfying (1.25) and (1.26) but not (1.19).

Theorem 1.4 only asserts the existence and value of the improper integral (1.29); here again, (1.20) also follows from Theorem 1.1. It is not known whether (1.9) is a consequence of (1.25) and (1.26). It is also open whether (1.26) can be replaced by the weaker condition (1.3) in (1.29) of Theorem 1.4.

The proof of (1.27) employs Theorem 1.5 below which is concerned with the nonlinear equation

$$x(t) + \int_0^t a(t - s) g(x(s))\, ds = f(t), \tag{1.30}$$

where a, g, and f are prescribed.

THEOREM 1.5. *Let* $f \in C[0, \infty) \cap BV_{\text{loc}}[0, \infty)$ *and let* (1.25), (1.26) *and the following conditions be satisfied*:

$$g \in C(-\infty, \infty), \quad \operatorname{meas}\{x < 0 \mid g(x) > 0\} \leqslant X < \infty, \tag{1.31}$$

$$\operatorname{meas}\{x > 0 \mid g(x) < 0\} \leqslant X. \tag{1.32}$$

Then (1.30) *has a continuous solution on* $[0, \infty)$. *Moreover, if* $x \in C[0, \infty)$ *is a solution of* (1.30), *then*

$$\max_{0 \leqslant s \leqslant t} |x(s)| \leqslant X + \min_{0 \leqslant s \leqslant t} |f(s)| + V(f, [0, t]) \qquad (0 \leqslant t < \infty), \tag{1.33}$$

where $V(f, [0, t])$ *is the total variation of* f *on* $[0, t]$.

Although the statement of Theorem 1.5 is slightly stronger than that of Theorem 1 of Levin [3], it is proven in exactly the same manner as the latter result. Corollaries 1 and 2 of [3] treat the problem of replacing (1.26) with (1.3) in Theorem 1.5.

The proof of (1.28) employs:

LEMMA 1.3. *Let* (1.3), (1.25), *and*

$$f \in C[0, \infty), \quad f \textit{ is nonnegative and nondecreasing on } [0, \infty) \tag{1.34}$$

hold. Then the solution, x, *of* (1.1) *satisfies*

$$0 \leqslant x(t) \leqslant f(t) \qquad (0 \leqslant t < \infty). \tag{1.35}$$

A strengthened form of Lemma 1.3 is noted in §5.

The proof of the other assertions of Theorem 1.4 employ (1.28) and some results of Levin and Shea [4], concerning the asymptotic behavior of the bounded solutions of various integral equations, which are stated in §3. In the remainder of this Introduction we discuss Lemma 1.3 and, in particular, two quite different theorems dealing with nonlinear equations which are related to it.

An elementary self-contained proof of Lemma 1.3 is given in §2. With some rather obvious modifications, the arguments of §2 can be extended to (1.30) under the hypothesis

$$g \in C(-\infty, \infty), \quad xg(x) \geqslant 0 \qquad (-\infty < x < \infty). \tag{1.36}$$

We omit the details since the resulting generalization of Lemma 1.3 is essentially contained, as a quite special case, in Theorems 1.6 and 1.7 below. Observe that if (1.36) holds then so do (1.31) and (1.32) with $X = 0$.

The next result, which concerns the nonlinear equation

$$x(t) + \int_0^t a(t-s)\, g(x(s), s)\, ds = f(t), \tag{1.37}$$

is due to Friedman [1] and is discussed and employed by Miller [5] in his proof of Theorem 1.2 above. A related earlier result, for $a(t) = t^{-1/2}$, is due to Padmavally [9].

THEOREM 1.6. *Let* (1.3) *and the following conditions hold*:

$$a \in C(0, \infty), \quad f \in C[0, \infty), \quad a(t) > 0, \quad f(t) > 0 \qquad (0 < t < \infty),$$

$$\frac{f(T)}{f(t)} \leqslant \frac{a(T-s)}{a(t-s)} \qquad (0 \leqslant s \leqslant T < t),$$

g is measurable and $xg(x, t) \geqslant 0$ on $(-\infty, \infty) \times [0, \infty)$,

$g(\cdot, t) \in C(-\infty, \infty)$ for each $t \in [0, \infty)$.

If a, g, and f are sufficiently smooth to guarantee the uniqueness of the solution of (1.37), *then the latter exists on* $[0, \infty)$ *and satisfies* (1.35).

Lemma 1.3 may be obtained from Theorem 1.6 by setting $g(x, t) = x$ in (1.37) and reasoning as follows: Let $a > 0$ and $f > 0$ satisfy the hypothesis of Lemma 1.3. Then, obviously,

$$\frac{f(T)}{f(t)} \leqslant 1 \leqslant \frac{a(T-s)}{a(t-s)} \qquad (0 \leqslant s \leqslant T < t).$$

If $a \in C(0, \infty)$ is also assumed, then Theorem 1.6 asserts that the solution of (1.1) satisfies (1.35). The continuity and positivity restrictions are easily removed by approximation arguments of the sort given in §2.

A lower bound is obtained on the solutions of (1.30) in:

THEOREM 1.7. *Let* (1.25), (1.26), (1.31), *and* (1.34) *hold and let* $x \in C[0, \infty)$ *be a solution of* (1.30). *Then*

$$-X \leqslant x(t) \qquad (0 \leqslant t < \infty). \tag{1.38}$$

That Lemma 1.3 is a consequence of Theorem 1.7 may be seen as follows: Let (1.26) and the hypothesis of Lemma 1.3 hold. Then, by Theorem 1.7 with $X = 0$, $x(t) \geqslant 0$. This together with our assumptions implies that $x(t) \leqslant f(t)$, establishing Lemma 1.3 when (1.26) holds. The latter restriction may be removed by an approximation argument given in §2.

The proof of Theorem 1.7 is similar to that of Theorem 1.5. However, the argument is sufficiently intricate, and the changes required in [3] sufficiently numerous, to warrant giving a self-contained treatment of one of the subcases that make up its proof. This is done in §4.

Theorems 1.5 and 1.7 obviously imply:

COROLLARY. *Let* (1.32) *and the hypothesis of Theorem* 1.7 *hold. Then*

$$-X \leqslant x(t) \leqslant X + f(t) \qquad (0 \leqslant t < \infty).$$

2. **Proof of Lemma 1.3.** Approximation arguments will enable us to deduce Lemma 1.3 from the weaker, but easily proven, result:

LEMMA 2.1. *Let*

$$a \in C^1[0, \infty), \quad a(t) \geqslant 0, \quad a'(t) \leqslant 0 \qquad (0 \leqslant t < \infty), \tag{2.1}$$

$$f \in C^1[0, \infty), \quad f(t) > 0, \quad f'(t) \geqslant 0 \qquad (0 \leqslant t < \infty). \tag{2.2}$$

Then the solution, x, *of* (1.1) *satisfies*

$$0 < x(t) \leqslant f(t) \qquad (0 \leqslant t < \infty). \tag{2.3}$$

Suppose

$$0 < x(t) \qquad (0 \leqslant t < \infty) \tag{2.4}$$

does not hold. Then, since $x(0) = f(0) > 0$, there exists a unique $t_0 > 0$ such that

$$x(t_0) = 0, 0 < x(t) \qquad (0 \leqslant t < t_0). \tag{2.5}$$

This implies, as $x \in C^1[0, \infty)$, that

$$x'(t_0) \leqslant 0. \tag{2.6}$$

Differentiating (1.1), setting $t = t_0$, and invoking (2.2) yields

$$\begin{aligned} x'(t_0) &= f'(t_0) - \int_0^{t_0} a'(t_0 - s)x(s)\,ds \\ &\geqslant -\int_0^{t_0} a'(t_0 - s)x(s)\,ds. \end{aligned} \tag{2.7}$$

From (2.1), (2.5), and (2.7) it follows that $x'(t_0) > 0$, which contradicts (2.6), unless

$$a(t) \equiv a(0) \qquad (0 \leqslant t \leqslant t_0). \tag{2.8}$$

However, (2.8) and (1.1) obviously imply

$$x(t) + a(0)\int_0^t x(s)\,ds = f(t) \qquad (0 \leqslant t \leqslant t_0). \tag{2.9}$$

From (2.9) and (2.2) we easily obtain

$$x(t_0) \geqslant f(0)e^{-a(0)t_0} > 0,$$

which contradicts (2.5). Thus (2.4) is established, which together with the hypothesis clearly implies (2.3) and completes the proof of Lemma 2.1.

Turning to the proof of Lemma 1.3 itself, we let x_ε be the solution of

$$x_\varepsilon(t) + \int_0^t a(t - s + \varepsilon)x_\varepsilon(s)\,ds = f(t) + \varepsilon \qquad (0 \leqslant t < \infty), \tag{2.10}$$

where $\varepsilon > 0$ and a and f satisfy the hypothesis of Lemma 1.3. If (1.26) holds, $a(t - s + \varepsilon)$ may, but need not, be replaced by $a(t - s)$ in (2.10). If $f(0) > 0$, $f(t) + \varepsilon$ may, but need not, be replaced by $f(t)$ in (2.10). Such changes require setting $\varepsilon = 0$ in appropriate formulas below. We now show that

$$0 \leqslant x_\varepsilon(t) \leqslant f(t) + \varepsilon \qquad (0 \leqslant t < \infty). \tag{2.11}$$

If $\varepsilon = 0$ throughout (2.10), then $\varepsilon = 0$ in (2.11). Thus, in this special case, establishing (2.11) will complete the proof.

Let $\{a_n(\cdot, \varepsilon)\}$ satisfy

$$\begin{gathered} a_n(\cdot, \varepsilon) \in C^\infty[0, \infty), \quad a_n(0, \varepsilon) = a(\varepsilon), \quad a_n(\cdot, \varepsilon) \text{ is nonincreasing,} \\ a(t + \varepsilon) \leqslant a_{n+1}(t, \varepsilon) \leqslant a_n(t, \varepsilon) \qquad (0 \leqslant t < \infty), \\ \lim_{n\to\infty} a_n(t, \varepsilon) = a(t + \varepsilon) \quad \text{a.e.} \end{gathered} \tag{2.12}$$

for each $\varepsilon > 0$ and let $\{f_n\}$ satisfy

$$\begin{gathered} f_n \in C^\infty[0, \infty], \quad f_n(0) = f(0), \quad f_n \text{ is nondecreasing,} \\ f(t) \geqslant f_{n+1}(t) \geqslant f_n(t), \quad \lim_{n\to\infty} f_n(t) = f(t) \qquad (0 \leqslant t < \infty). \end{gathered} \tag{2.13}$$

If

$$\begin{gathered} \beta\colon [0, 1] \to [0, 1], \qquad \beta \in C^\infty[0, 1], \\ \beta(t) \equiv 1 \quad \left(0 \leqslant t \leqslant \tfrac{1}{3}\right), \qquad \beta'(t) \leqslant 0 \quad \left(\tfrac{1}{3} \leqslant t \leqslant \tfrac{2}{3}\right), \\ \beta(t) \equiv 0 \quad \left(\tfrac{2}{3} \leqslant t \leqslant 1\right), \end{gathered}$$

and if for $n, k = 0, 1, \ldots$ we set

$$a_n(t, \varepsilon) = \left\{a\left(\frac{k-1}{3^n} + \varepsilon\right) - a\left(\frac{k}{3^n} + \varepsilon\right)\right\} \cdot \beta\left(3^n\left(t - \frac{k}{3^n}\right)\right) + a\left(\frac{k}{3^n} + \varepsilon\right) \tag{2.14}$$

on $k/3^n \leqslant t < (k+1)/3^n$, where $a(-t+\varepsilon) = a(\varepsilon)$ for $t \geqslant 0$, then it is not hard to show that the a_n of (2.14) satisfy (2.12). A sequence satisfying (2.13) may be similarly constructed.

Let $y_n(\cdot, \varepsilon)$ be the solution of

$$y_n(t, \varepsilon) + \int_0^t a_n(t-s, \varepsilon) y_n(s, \varepsilon)\, ds = f_n(t) + \varepsilon \qquad (0 \leqslant t < \infty).$$

Lemma 2.1 implies that

$$0 < y_n(t, \varepsilon) \leqslant f_n(t) + \varepsilon \qquad (0 \leqslant t < \infty). \tag{2.15}$$

It is an elementary matter using the Ascoli-Arzelà lemma and the uniqueness of $x_\varepsilon(t)$ to show that

$$\lim_{n \to \infty} y_n(t, \varepsilon) = x_\varepsilon(t) \tag{2.16}$$

uniformly on $[0, T]$ for every $T > 0$. Clearly (2.11) is an immediate consequence of (2.13), (2.15), and (2.16).

In a similar manner, setting $\varepsilon = 1/m$ $(m = 1, 2, \ldots)$ in (2.10) and (2.11) and letting $m \to \infty$ yields

$$\lim_{m \to \infty} x_{1/m}(t) = x(t) \tag{2.17}$$

uniformly on $[0, T]$ for every $T > 0$, where x is the solution of (1.1). The conclusion of Lemma 1.3 now follows from (2.11) and (2.17).

3. **Proof of Theorem 1.4.** We consider (1.27) first. Let (1.25) and (1.26) hold and let $a_n(t) = a_n(t, 0)$, where $a_n(t, \varepsilon)$ are defined by (2.14). Then (2.12) implies

$$\begin{gathered} a_n \in C^\infty[0, \infty), \quad a_n(0) = a(0), \quad a_n \text{ is nonincreasing,} \\ a(t) \leqslant a_{n+1}(t) \leqslant a_n(t) \qquad (0 \leqslant t < \infty), \\ \lim_{n \to \infty} a_n(t) = a(t) \quad \text{a.e.} \end{gathered} \tag{3.1}$$

Let r_n be the solution of

$$r_n(t) + \int_0^t a(t-s) r_n(s)\, ds = a_n(t) \qquad (0 \leqslant t < \infty). \tag{3.2}$$

This equation is the special case of (1.30) in which

$$x \to r_n, \quad a = a, \quad g(x) = x \to r_n, \quad X = 0, \quad f \to a_n. \tag{3.3}$$

From (3.1) we have

$$\min_{0 \leq s \leq t} |a_n(s)| + V(a_n, [0, t]) = a(0). \tag{3.4}$$

In view of (3.3) and (3.4), applying Theorem 1.5 to (3.2) yields

$$-a(0) \leq r_n(t) \leq a(0) \qquad (0 \leq t < \infty; n = 1, 2, \ldots). \tag{3.5}$$

Let

$$z_n(t) = \int_0^t a(t-s) r_n(s)\, ds. \tag{3.6}$$

It is easily verified that (1.25), (1.26), (3.5), and (3.6) imply

$$|z_n(t)| \leq a^2(0)t, \quad |z_n(t+\Delta) - z_n(t)| \leq 2a^2(0)\Delta \qquad (0 \leq t, \Delta < \infty)$$

for $n = 1, 2, \ldots$. A routine exercise involving the Ascoli-Arzelà lemma, the Lebesgue bounded convergence theorem, and the uniqueness of the solution of (1.2) now shows that $r_n(t) \to r(t)$ a.e. $(n \to \infty)$ and (1.27) are satisfied.

Let

$$A(t) = \int_0^t a(s)\, ds, \quad R(t) = \int_0^t r(s)\, ds, \quad V(t) = 1 - R(t). \tag{3.7}$$

Then $R, V \in AC_{\mathrm{loc}}[0, \infty)$. Integrating (1.2) yields

$$R(t) + \int_0^t a(t-s) R(s)\, ds = A(t) \qquad (0 \leq t < \infty), \tag{3.8}$$

which together with (3.7) implies

$$V(t) + \int_0^t a(t-s) V(s)\, ds = 1 \qquad (0 \leq t < \infty). \tag{3.9}$$

Applying Lemma 1.3 to (3.9) yields

$$0 \leq V(t) \leq 1 \qquad (0 \leq t < \infty) \tag{3.10}$$

and, because of (3.7),

$$0 \leq R(t) \leq 1 \qquad (0 \leq t < 1), \tag{3.11}$$

which establishes (1.28) of the conclusion. It may be noted from the proof that (1.28) holds with (1.26) replaced by (1.3).

The equations

$$x(t) + \int_0^t x(t-s)\, dB(s) = z(t) \qquad (0 \leq t < \infty), \tag{3.12}$$

$$x'(t) + \int_0^t x(t-s)\, dB(s) = z(t) \qquad (0 \leq t < \infty), \tag{3.13}$$

where

$$B \in BV[0, \infty), \quad B(0) = 0, \quad B(t-) = B(t) \qquad (0 < t < \infty), \tag{3.14}$$

$$z \in L^\infty(0, \infty), \quad \lim_{t \to \infty} z(t) = z(\infty) \text{ exists}, \tag{3.15}$$

are employed in the remainder of this proof. Let

$$\hat{B}(\lambda) = \int_0^\infty e^{-i\lambda t}\, dB(t) \qquad (-\infty < \lambda < \infty), \tag{3.16}$$

$$S_1(B) = \{\lambda | \hat{B}(\lambda) = -1, -\infty < \lambda < \infty\}, \tag{3.17}$$

$$S_2(B) = \{\lambda | \hat{B}(\lambda) = -i\lambda, -\infty < \lambda < \infty\}. \tag{3.18}$$

Concerning (3.12) and (3.13) we employ, respectively, the following immediate and very special consequences of Theorems 3b and 3a and Lemmas 2.3 and 2.2 of Levin and Shea [4]:

LEMMA 3.1. *Let* (3.14) *and* (3.15) *hold and let* $x \in L^\infty(0, \infty)$ *be Borel measurable and satisfy* (3.12) *and*

$$\lim_{\substack{t\to\infty \\ \tau\to 0}} |x(t+\tau) - x(t)| = 0. \tag{3.19}$$

Then $S_1(B) = \varnothing$ *implies*

$$\lim_{t\to\infty} x(t) = \frac{z(\infty)}{1 + B(\infty)}. \tag{3.20}$$

LEMMA 3.2. *Let* (3.14) *and* (3.15) *hold, with* $z(\infty) = 0$, *and let* $x \in AC_{\text{loc}}[0, \infty) \cap L^\infty(0, \infty)$ *satisfy* (3.13) *a.e. Then*

$$x(t) = c(t) + \eta(t) \qquad (0 \leqslant t < \infty), \tag{3.21}$$

where

$$\lim_{t\to\infty} \eta(t) = 0, \qquad \lim_{t\to\infty} \left\{ \operatorname*{ess\,sup}_{t<s<\infty} |\eta'(s)| \right\} = 0, \tag{3.22}$$

and

$$S_2(B) = \varnothing \text{ implies } c(t) \equiv 0, \tag{3.23}$$

$$\begin{gathered} S_2(B) = \{0\} \text{ implies } c \in C^\infty[0, \infty) \cap L^\infty(0, \infty), \\ \lim_{t\to\infty} c^{(j)}(t) = 0 \qquad (j \geqslant 1). \end{gathered} \tag{3.24}$$

The cases

$$\int_0^\infty a(t)\, dt < \infty, \tag{3.25}$$

$$\int_0^\infty a(t)\, dt = \infty \tag{3.26}$$

are treated separately in the remainder of this proof.

If (3.25) holds, we employ (3.7) and write (3.9) as

$$V(t) + \int_0^t V(t-s)\, dA(s) = 1. \tag{3.27}$$

From (3.7) and (3.16) we have

$$\hat{A}(\lambda) = \int_0^\infty a(t)\cos\lambda t\,dt - i\lambda\int_0^\infty a(t)\sin\lambda t\,dt,$$

which together with (1.25) and (3.25) ((1.26) is not required here) implies that $S_1(A) = \varnothing$. From (3.9), (3.10), and (3.25) it follows that V is uniformly continuous on $[0, \infty)$ and, hence, satisfies the Tauberian condition (3.19). Therefore, applying Lemma 3.1 to (3.27) yields $V(\infty) = [1 + A(\infty)]^{-1}$, which in view of (3.7) establishes (1.29) and (1.20) for the case (3.25).

When (3.26) holds we define B_1 by

$$B_1(0) = 0, \quad B_1(t) = a(t) \qquad (0 < t < \infty) \tag{3.28}$$

and assume, without loss of generality, that

$$a(t-) = a(t) \qquad (0 < t < \infty), \quad a(0) = a(0+). \tag{3.29}$$

Then, by (1.25), (1.26), (3.28), and (3.29), B_1 satisfies (3.14) and

$$\hat{B}_1(\lambda) = a(0) + \int_0^\infty \cos\lambda t\,da(t) - i\int_0^\infty \sin\lambda t\,da(t). \tag{3.30}$$

From (1.25), (1.26), and (3.30) it is easily seen that

$$a(\infty) > 0 \text{ implies } S_2(B_1) = \varnothing, \tag{3.31}$$

$$a(\infty) = 0 \text{ implies } S_2(B_1) = \{0\}. \tag{3.32}$$

Differentiating (3.9) yields

$$V'(t) + \int_0^t V(t-s)\,dB_1(s) = 0 \quad \text{a.e. on } [0, \infty), \tag{3.33}$$

to which, in view of (3.7) and (3.10), we can apply Lemma 3.2. If $a(\infty) > 0$, then (3.31) and (3.21)–(3.23) imply

$$V(\infty) = 0. \tag{3.34}$$

From (3.34) and (3.7) we have $R(\infty) = 1$, which establishes (1.29) and (1.21) in this subcase of (3.26).

If $a(\infty) = 0$ and (3.26) hold, then (3.32) and Lemma 3.2 imply

$$V(t) = c(t) + \eta(t) \qquad (0 \leqslant t < \infty) \tag{3.35}$$

where c and η satisfy (3.24) and (3.22) respectively. Suppose (3.34) does not hold. Then (3.10), (3.22), (3.24), and (3.35) yield the existence of sequences $\{t_n\}$ and $\{T_n\}$ and a constant $\nu > 0$ such that

$$\lim_{n\to\infty} t_n = \infty, \quad \lim_{n\to\infty} T_n = \infty, \quad V(t) \geqslant \nu \qquad (t_n \leqslant t \leqslant t_n + T_n). \tag{3.36}$$

From (1.25), (3.9), (3.10), and (3.36) it follows that

$$\nu\int_0^{T_n} a(s)\,ds \leqslant 1 \qquad (n = 1, 2, \ldots),$$

which contradicts (3.26) for sufficiently large n. Hence (3.34) holds and, similar to the preceding paragraph, (1.29) and (1.21) are also established in this subcase of (3.26). This completes the proof of Theorem 1.4.

4. **Proof of Theorem 1.7.** Let x be a solution of (1.30) and let

$$P = \{t \mid g(x(t)) > 0\}, \qquad Q = \{t \mid g(x(t)) < 0\}, \qquad R = \{t \mid g(x(t)) = 0\}. \tag{4.1}$$

$$p(t) = \int_0^t (g(x(s)))^+ a(t-s)\,ds, \qquad q(t) = \int_0^t (g(x(s)))^- a(t-s)\,ds, \tag{4.2}$$

where $y^+ = \max(y, 0)$, $y^- = \max(-y, 0)$. Then $p \geqslant 0$, $q \geqslant 0$ and

$$x(t) + p(t) - q(t) = f(t) \qquad (0 \leqslant t < \infty). \tag{4.3}$$

Lemma 1 of [3] implies that $p, q \in AC_{\text{loc}}[0, \infty)$ and

$$p'(t) \leqslant 0 \quad \text{a.e. on } Q \cup R, \qquad q'(t) \leqslant 0 \quad \text{a.e. on } P \cup R. \tag{4.4}$$

From (1.34), (4.3), and $p, q \in AC_{\text{loc}}[0, \infty)$ it follows that $x \in C[0, \infty) \cap BV_{\text{loc}}[0, \infty)$ and that each of the infinite series appearing below converge absolutely.

Let $t_0 \in [0, \infty)$. In showing that

$$x(t_0) \geqslant -X, \tag{4.5}$$

we analyze each of the following situations separately:

$$0 \in Q \cup R, \qquad t_0 \in Q \cup R, \tag{4.6}$$

$$0 \in Q \cup R, \qquad t_0 \in P, \tag{4.7}$$

$$0 \in P, \qquad t_0 \in Q \cup R, \tag{4.8}$$

$$0 \in P, \qquad t_0 \in P, \qquad [0, t_0) \not\subset P, \tag{4.9}$$

$$0 \in P, \qquad t_0 \in P, \qquad [0, t_0) \subset P. \tag{4.10}$$

The hypothesis and (4.1)–(4.3) easily imply that (4.5) is a consequence of (4.10). For brevity we will only prove here that (4.6) implies (4.5). The arguments for the remaining cases, (4.7)–(4.9), may be readily constructed from those that follow and from the discussion in [3].

Thus, let (4.6) hold. Then (4.1) implies

$$P = \bigcup_{k=1}^{N_1} (\alpha_k', \alpha_k'') \qquad (\alpha_k', \alpha_k'' \in R),$$

where $N_1 \geqslant 1$ or ∞ if $P_1 \neq \varnothing$, $N_1 = 0$ if $P = \varnothing$, and there may exist a k_1'' such that $\alpha_{k_1}'' = \infty$. Let

$$P_{t_0} = P \cap [0, t_0], \qquad I = \{k \geq 1 | \alpha_k'' \leq t_0\}.$$

Then

$$P_{t_0} = \bigcup_{k \in I} (\alpha_k', \alpha_k''). \tag{4.11}$$

As a step toward proving (4.5), (4.3) suggests obtaining an upper bound on $p(t_0)$. From (4.1)–(4.4) and (4.11) we have

$$\begin{aligned} p(t_0) &= \int_0^{t_0} p'(s)\, ds \leq \int_{P_{t_0}} p'(s)\, ds \\ &= \sum_{k \in I} \int_{\alpha_k'}^{\alpha_k''} p'(s)\, ds = \sum_{k \in I} [p(\alpha_k'') - p(\alpha_k')] \\ &= \sum_{k \in I} \{[x(\alpha_k') - x(\alpha_k'')] + [q(\alpha_k'') - q(\alpha_k')] + [f(\alpha_k'') - f(\alpha_k')]\} \\ &\leq \sum_{k \in I} [x(\alpha_k') - x(\alpha_k'')] + \sum_{k \in I} [f(\alpha_k'') - f(\alpha_k')]. \end{aligned} \tag{4.12}$$

In order to study the term $\sum_{k \in I}[x(\alpha_k') - x(\alpha_k'')]$, let

$$A = \{x | g(x) > 0\}, \qquad C = \{x | g(x) = 0\}.$$

Then

$$A = \bigcup_{i=1}^{N_2} (\xi_i', \xi_i'') \qquad (\xi_i', \xi_i'' \in C),$$

where $N_2 \geq 1$ or ∞ if $A \neq \varnothing$, $N_2 = 0$ if $A = \varnothing$, and there may exist an i_1'' such that $\xi_{i_1}'' = \infty$. Let $I = (c_1) \cup (c_2) \cup (c_3)$ where

$$(c_1) = \{k \in I | x(\alpha_k') < x(\alpha_k'')\},$$
$$(c_2) = \{k \in I | x(\alpha_k') > x(\alpha_k'')\},$$
$$(c_3) = \{k \in I | x(\alpha_k') = x(\alpha_k'')\}.$$

Then for each $k \in I$ there exists a $\phi(k)$ such that

$$k \in (c_1) \text{ implies } x(\alpha_k') = \xi_{\phi(k)}', \quad x(\alpha_k'') = \xi_{\phi(k)}'',$$
$$k \in (c_2) \text{ implies } x(\alpha_k') = \xi_{\phi(k)}'', \quad x(\alpha_k'') = \xi_{\phi(k)}',$$
$$k \in (c_3) \text{ implies } x(\alpha_k') = x(\alpha_k'') = \xi_{\phi(k)}' \quad \text{or} \quad x(\alpha_k') = x(\alpha_k'') = \xi_{\phi(k)}'',$$

where $\phi(k)$ is uniquely defined for $k \in (c_1) \cup (c_2)$ but may be double valued when $k \in (c_3)$. Clearly

$$\sum_{k \in I} [x(\alpha_k') - x(\alpha_k'')] = \sum_{k \in I - (c_3)} [x(\alpha_k') - x(\alpha_k'')]. \tag{4.13}$$

Let

(4.14) $\qquad I_1(i) = \{k | k \in I - (c_3), \phi(k) = i\} \qquad (i = 1, 2, \ldots).$

Thus $I_1(i)$ is a finite or empty set. It is evident from the graph of x that for each $i \geq 1$:

$$\sum_{k \in I_1(i)} [x(\alpha_k') - x(\alpha_k'')] = \begin{cases} \xi_i' - \xi_i'' & \text{if } x(0) \leq \xi_i' < \xi_i'' \leq x(t_0), \\ \xi_i'' - \xi_i' & \text{if } x(t_0) \leq \xi_i' < \xi_i'' \leq x(0), \\ 0 & \text{if } x(0), x(t_0) \leq \xi_i' \text{ or} \\ & \qquad \xi_i'' \leq x(0), x(t_0). \end{cases} \tag{4.15}$$

Let

$$I_2 = \{i \geq 1 | x(t_0) \leq \xi_i' < \xi_i'' \leq x(0)\}. \tag{4.16}$$

From (4.13)–(4.16) and (1.31) it follows that

$$\begin{aligned} \sum_{k \in I} [x(\alpha_k') - x(\alpha_k'')] &= \sum_{i=1}^{\infty} \sum_{k \in I_1(i)} [x(\alpha_k') - x(\alpha_k'')] \\ &\leq \sum_{i \in I_2} \sum_{k \in I_1(i)} [x(\alpha_k') - x(\alpha_k'')] \\ &= \sum_{i \in I_2} (\xi_i'' - \xi_i') \leq f(0) + X. \end{aligned} \tag{4.17}$$

From (4.3), (4.12), (4.17), and $q(t_0) \geq 0$ we have

$$x(t_0) \geq -X - f(0) + \sum_{k \in I} [f(\alpha_k') - f(\alpha_k'')] + f(t_0),$$

which together with (1.34) obviously implies (4.5) and completes the proof.

5. **Supplementary remarks.** It easily follows from Lemma 1.3 and an approximation argument of the type employed in §2 (note (2.13)) that

(5.1) $\qquad$ Lemma 1.3 is valid without the hypothesis $f \in C[0, \infty)$.

From (1.2) and (3.9) it follows that

$$a(0)V(t) - r(t) + \int_0^t a(t-s)(a(0)V(s) - r(s))\, ds = a(0) - a(t),$$

which together with (1.25), (1.26), and (5.1) implies

$$0 \leq a(0)V(t) - r(t) \leq a(0) - a(t). \tag{5.2}$$

Combining (3.7) and (5.2) yields

$$\begin{aligned} a(t) - a(0)\int_0^t r(s)\, ds &\leq r(t) \\ &\leq a(0)\left(1 - \int_0^t r(s)\, ds\right) \qquad (0 \leq t < \infty). \end{aligned} \tag{5.3}$$

From (1.28) we see that (5.3) is stronger than (1.27). When integrated, (5.3) implies

$$-a(0)(1-e^{-a(0)t})+a(t) \leqslant r(t) \leqslant a(0)-a(0)\int_0^t e^{-a(0)(t-s)}a(s)\,ds.$$

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